A new algorithm of evaluation of Wigner Distribution Function with application to radio occultation analysis

Michael Gorbunov[1,2,3], Oksana Koval[1]

[1] A.M.Obukhov Institute of Atmospheric Physics,
[2] Spire Global, Inc.,
[3] Hydrometeorological Research Center of Russian Federation

Introduction

Time-frequency representation is a powerful technique for the analysis of radio occultation (RO) data. It includes sliding-spectrum analysis, or spectrogram, Wigner Distribution Function (WDF), Kirkwood Distribution Function (KDF), and their modifications based on the reassignment technique. Using these techniques, a signal as a 1-D function of time is mapped to 2-D phase or ray space, where each point can be associated with a geometric optical ray. This space is parameterized by coordinate and momentum. Two possible parameterizations: (time – Doppler frequency) and (impact parameter – bending angle) are linked to each other by a canonical transform. WDF demonstrates a higher resolution as compared to spectrogram and more regular behavior as compared to KDF. On the other hand, the evaluation of WDF requires high computational costs, while the evaluation of KDF is fast.

In this study, we apply the theory of WDF, KDF, and Fourier integral operators (FIOs) to develop a fast algorithm of the evaluation of WDF. We employ the fact that between WDF and KDF there are forward and inverse convolution relations. We consider the rotation group of the phase space. This group consists of canonical transforms, which are implemented as FIOs describing the dynamics of the harmonic oscillator. Such FIOs form the group of fractional Fourier transforms. The fundamental property of WDF is its invariance with respect to canonical transforms. KDF does not possess this property. However, we consider KDF averaged over the rotation group. Because KDF equals WDF convolved with the corresponding kernel, its straightforward to arrive at the expression of the KDF averaged over the rotation group. WDF is invariant, and, therefore, the averaged KDF equals WDF convolved with the averaged convolution kernel. The latter is explicitly expressed through Bessel function. Finally, this procedure results in WDF smoothed over a quantum cell in the phase space.

We implemented the algorithm numerically. The fractional Fourier transform can be represented as a composition of multipliers and Fourier transforms, which means that it allows a fast numerical implementation. We perform averaging over a finite number of rotation angles, which allows the performance optimization. We give examples of processing artificial and real RO events.

Wigner and Kirkwood Distribution Functions

Wigner Distribution Function (WDF):

$$\rho^W(x,\xi) = \frac{k}{2\pi}\int \psi\left(x-\frac{s}{2}\right)\bar{\psi}\left(x+\frac{s}{2}\right)\exp(iks\xi)\,ds = \\ = \int \bar{\tilde{\psi}}\left(\xi-\frac{\sigma}{2}\right)\tilde{\psi}\left(\xi+\frac{\sigma}{2}\right)\exp(ikx\sigma)\,d\sigma\,,$$

where x is the [illegible], ξ is the corresponding frequency, k is wavenumber, and $\psi(x)$ is the observed wave field.

Kirkwood Distribution Function (KDF):

$$\rho^{x\xi}(x,\xi) = \frac{k}{2\pi}\psi(x)\int \bar{\psi}(x+s)\exp(iks\xi)\,ds = \\ = \sqrt{\frac{-ik}{2\pi}}\psi(x)\exp(-ikx\xi)\ \sqrt{\frac{ik}{2\pi}}\int \bar{\psi}(\tilde{x})\exp(ik\tilde{x}\xi)\,d\tilde{x}\,.$$

KDF Properties

1. KDF is a complex, sign-alternating function (WDF is a real sign-alternating function).
2. Positive projections (similar to WDF):
$$\int \rho^{x\xi}(x,\xi)dx = \left|\tilde{\psi}(\xi)\right|^2, \\ \int \rho^{x\xi}(x,\xi)d\xi = |\psi(x)|^2.$$
3. Positive full energy (similar to WDF):
$$\int \rho^{x\xi}(x,\xi)dx\ d\xi = E = \int |\psi(x)|^2 dx = \int \left|\tilde{\psi}(\xi)\right|^2 d\xi\,.$$
4. Global univalent phase:
$$\psi(x) = a(x)\exp\big(ikS(x)\big), \\ \tilde{\psi}(\xi) = \tilde{a}(\xi)\exp\left(ik\tilde{S}(\xi)\right) \\ \rho^{x\xi}(x,\xi) = a^{x\xi}(x,\xi)\exp\left(ikS^{x\xi}(x,\xi)\right), \\ S^{x\xi}(x,\xi) = S(x) - \tilde{S}(\xi) - x\xi.$$
5. Convolution relations between KDF and WDF:
$$\rho^W(x,\xi) = \rho^{x\xi}(x,\xi) * T^W_{x\xi}(x,\xi), \\ T^W_{x\xi}(x,\xi) = \frac{k}{\pi}\exp(2ik\ \xi\ x)\,, \\ \rho^{x\xi}(x,\xi) = \rho^W(x,\xi) * T^{x\xi}_W(x,\xi), \\ T^{x\xi}_W(x,\xi) = \frac{k}{\pi}\exp(-2ik\ \xi\ x)$$

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Fractional Fourier Transform

Consider a phase space rotation $(x,\xi) \longrightarrow (y,\eta)$ by angle α:

$$y = x\cos\alpha + \xi\sin\alpha\,, \\ \eta = -x\sin\alpha + \xi\cos\alpha\,,$$

The 1st and 2nd type generating function $S_2(y,x,\alpha)$ of this transform is evaluated as follows:

$$dS_1 = \eta\,dy + x\,d\xi, \\ dS_2 = \eta\,dy - \xi\,dx, \\ S_1(y,\xi,\alpha) = -\frac{\xi^2\sin\alpha - 2\xi y + y^2\sin\alpha}{2\cos\alpha}, \\ S_2(y,x,\alpha) = \frac{y^2\cos\alpha - 2xy + x^2\cos\alpha}{2\sin\alpha}.$$

The corresponding Fourier Integral Operator represented as a 1st and 2nd type operators, is written as follows:

$$\psi_\alpha(y) \equiv \hat{\Phi}_\alpha\psi_0 = \frac{1}{\sqrt{2\pi i\sin\alpha}}\int \exp\left(i\frac{y^2\cos\alpha - 2xy + x^2\cos\alpha}{2\sin\alpha}\right)\psi_0(x)\ dx = \\ = \frac{1}{\sqrt{-2\pi i\cos\alpha}}\int \exp\left(-i\frac{\xi^2\sin\alpha - 2\xi y + y^2\sin\alpha}{2\cos\alpha}\right)\tilde{\psi}_0(\xi)\ d\xi\,,$$

where $\tilde{\psi}_0(\xi)$ is the Fourier transform of $\psi_0(x)$.

The properties of Fractional Fourier Transform:

1. The group property:
$$\hat{\Phi}_\alpha \circ \hat{\Phi}_\beta = \hat{\Phi}_\beta \circ \hat{\Phi}_\alpha = \hat{\Phi}_{\alpha+\beta}, \\ \left(\hat{\Phi}_\alpha\right)^{-1} = \hat{\Phi}_{-\alpha}, \\ \hat{\Phi}_0 = \hat{I}.$$
2. The Fourier Transform implements the phase space rotation by $\pi/2$:
$$\hat{\Phi}_{\pi/2}\psi = \tilde{\psi}$$
3. The WDF invariance with respect to FrFT:
$$\rho^W_\alpha(y,\eta) = \frac{k}{2\pi}\int \psi_\alpha\left(y-\frac{s}{2}\right)\bar{\psi}_\alpha\left(y+\frac{s}{2}\right)\exp(is\eta)\,ds = \rho^W(x,\xi).$$
4. Operator $\hat{\Phi}_\alpha$ is the fractional power $2\alpha/\pi$ of the Fourier Transform.

The Evaluation of Smoothed WDF from KDF

The evaluation of KDF is computationally cheap as compared to WDF: WDF requires a large number of long-term Fourier transform; KDF requires just one. Using the properties of the FrFT and the convolution relations between WDF and KDF, we can write the following relations:

$$\rho^{x\xi}_\alpha(y,\eta) = \frac{k}{2\pi}\psi_\alpha(y)\int \bar{\psi}_\alpha(y+s)\exp(is\eta)\,ds \\ \frac{1}{2\pi}\int \rho^{x\xi}_\alpha(y(\alpha,x,\xi),\eta(\alpha,x,\xi))d\alpha = \rho^W(x,\xi) * \frac{1}{2\pi}\int T^W_{x\xi}\big(y(\alpha,x,\xi),\eta(\alpha,x,\xi)\big)d\alpha = \\ = \rho^W(x,\xi)\ * \frac{1}{\pi}J_0\left(\sqrt{x^2+\xi^2}\right).$$

This results in the algorithm of evaluation of WDF convolved with the window function:
1. Multiple application of FrFT to the wave field;
2. Evaluation of KDF for different rotation angles α;
3. Averaging $\rho^{x\xi}_\alpha(y(\alpha,x,\xi),\eta(\alpha,x,\xi))$ over angle α.
4. The number of different angles for averaging may be chosen relatively small.

Examples of KDF for Test Signals Subjected to FrFT

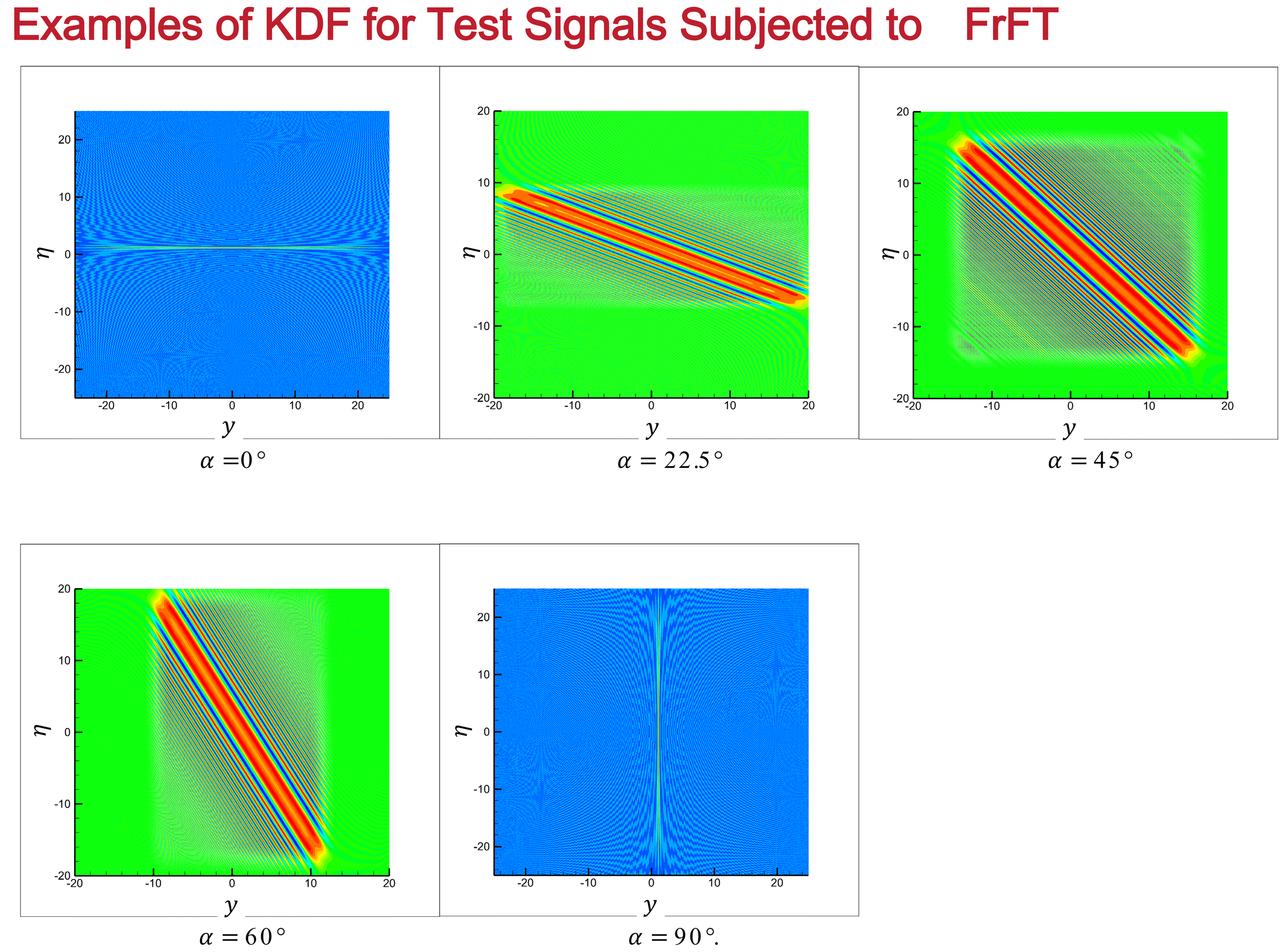

Figure 1: *Real part of KDF of apodized monochromatic test signal for* $\alpha = 0°, 22.5°, 45°, 60°$, and $90°$, in unitless coordinates (y,η).

Occultation Events Processing

KDF L1	KDF L2
WDF L1	WDF L2

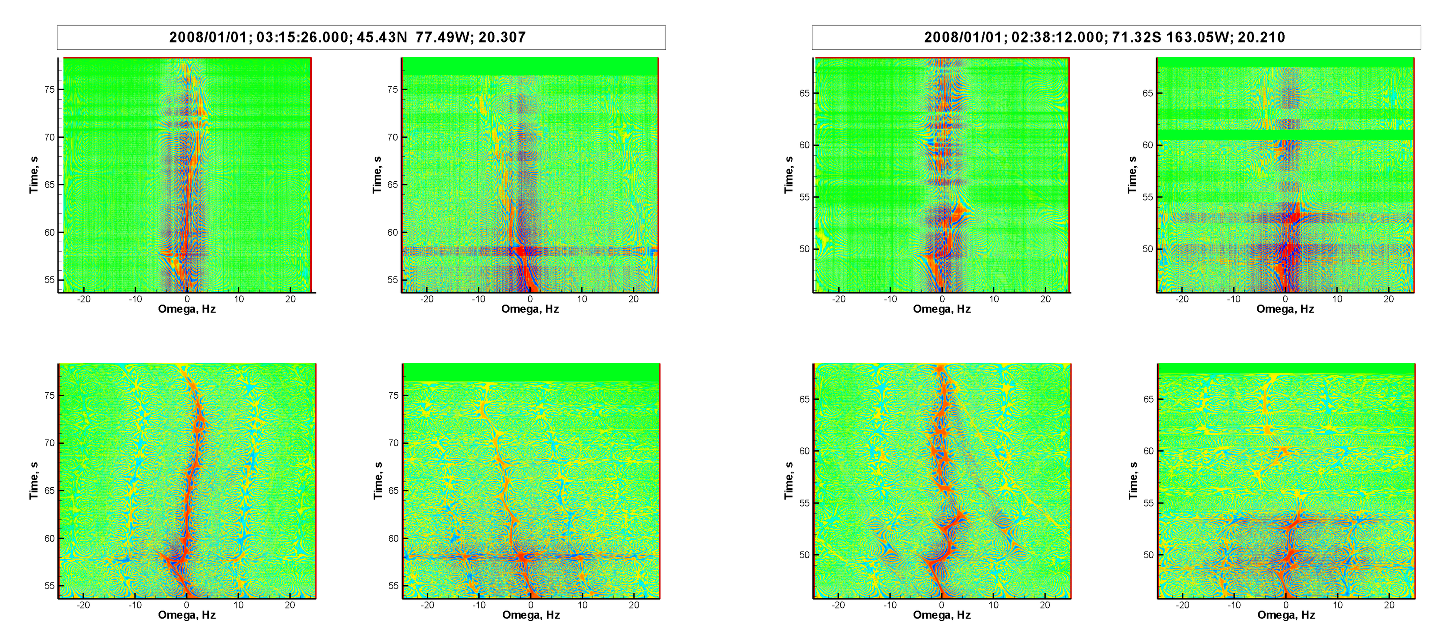

Figure 3: *Examples of KDF and WDF for COSMIC RO events*

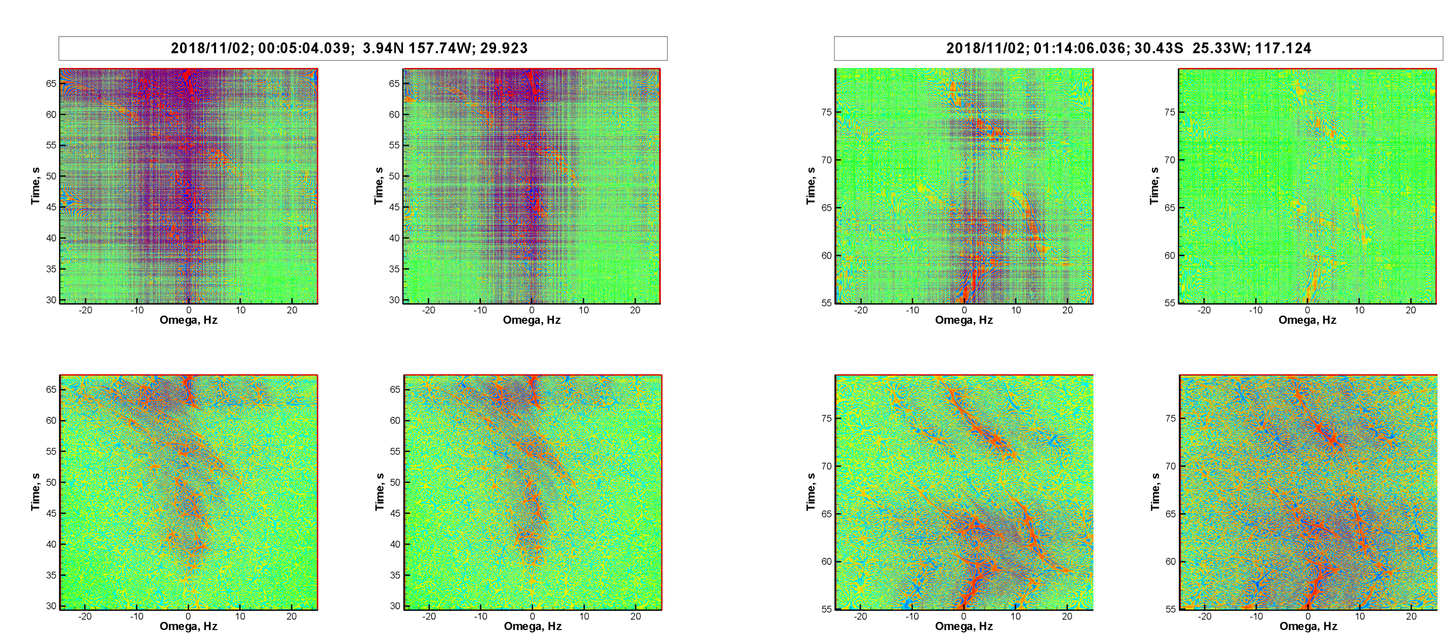

Figure 4: *Examples of KDF and WDF for Spire RO events. Despite the higher noise level in this figure, complicated multipath structures in the lower troposphere can be clearly identified in Spire WDF results. This is consistent with our observation that lower tropospheric statistics are very similar for Spire and CII results (see the poster by Irisov et al. at this Conference).*

Conclusion

- Wigner Distribution Function (WDF) and Kirkwood Distribution Function (WDF) are powerful means of the analysis of RO signal structure .
- WDF provides a better visualization, as compared to KDF. However, the evaluation of WDF is much more computationally expensive: it requires a large number of long-term Fourier Transforms .
- Using the convolution relations between WDF and KDF, and Fractional Fourier (FrFT) Transform Properties, we developed a new algorithm for the evaluation of WDF convolved with a smoothing window function . The algorithm requires a relatively small number of FrFTs, while FrFT is approximately as computationally expensive as the standard FT.

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m_e_gorbunov@mail.ru